

Using Power Series Methods to Model Dynamical Systems through Differential Equations

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Description of Dynamical Systems

Many dynamical systems can accurately described or better understood using differential equations. However, obtaining solutions to these equations can be difficult and many times impossible. Even if one obtains a closed form solution, it can be difficult to analyze. Also, in many cases, studying the differential equations can be of more value than studying the solution. Isaac Newton, one of the pioneers of differential equations who studied many dynamical systems realized that using polynomials to approximate the solutions of the differential equations gave him understanding of both the properties of the dynamical system and the solution. We studied four types of dynamical systems and used differential equations to model these dynamical systems. The four systems are networks, planetary motion, pressure disturbances in a medium caused by an explosion and the motion of particles comprising the medium disturbed by an explosion.

The differential equations used to model these systems are of the form

$$\mathbf{y}'(t) = F(\mathbf{y}(t)). \quad (1)$$

where t represents time and $\mathbf{y}$ represents the objects being studied.

Throughout history and from discovery, the idea of using polynomials to understand differential equations is known as power series methods (PSM), automatic differentiation (AD), differential transform methods (DTM) and modified Picard methods. We will use PSM because it better represents our work and ultimately includes all of these methods. In order to use PSM, we first converted the differential equation (1) to the form

$$\mathbf{u}'(t) = P(\mathbf{u}(t)), \quad (2)$$

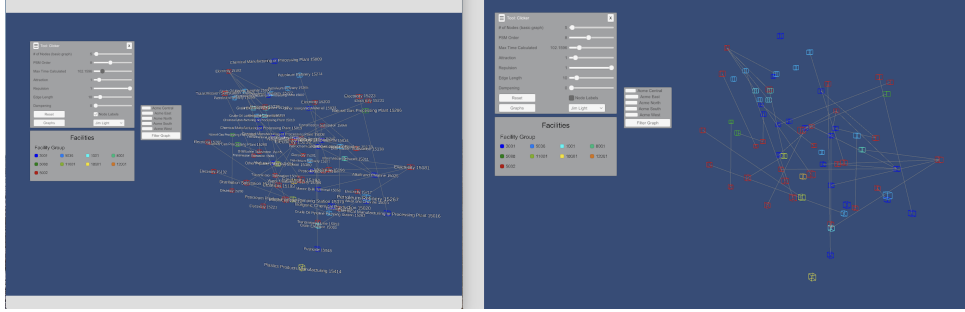
where P is a polynomial and the solution to (2) contains the solution to (1). Since P is a polynomial, one can obtain power series solutions to (2). When using the computer to obtain the power series, one can only obtain a finite number of the terms in the power series. That is, the computer can obtain a polynomial that approximates the power series solution to (2). The user of the computer program that generates the polynomial approximation chooses the number of terms to use because PSM is parameter free for this number. This allows the user to obtain as accurate a solution as desired.

The Four Dynamical Systems

We now describe the dynamical systems and the differential equations for our four systems

1. NETWORKS

(a) Force Directed Layout (FDL)



Two snapshots of an active FDL of substation in an infrastructure with and without labels for the substations.

FDL's use electrodynamic relations and attractive force relations to model the behavior of N particles that are grouped in various ways through the attractive force, but have an overall governing electrodynamic relation. The dynamics on each particle is electromotive force plus attractive force plus damping force and the differential equation modeling this is given by

$$\begin{aligned}
\mathbf{r}'_j(t) &= \mathbf{v}_j(t) \\
\mathbf{v}'_j(t) &= \sum_{n=1}^{j-1} \frac{q_n q_j}{\|\mathbf{r}_n - \mathbf{r}_j\|^3} (\mathbf{r}_n - \mathbf{r}_j) + \sum_{n=j+1}^N \frac{q_n q_j}{\|\mathbf{r}_n - \mathbf{r}_j\|^3} (\mathbf{r}_n - \mathbf{r}_j) + \\
&\quad \sum_{n=1}^{j-1} A_{n,j} (\mathbf{r}_n - \mathbf{r}_j) + \sum_{n=j+1}^N A_{n,j} (\mathbf{r}_n - \mathbf{r}_j) + D_j \mathbf{v}_j.
\end{aligned}$$

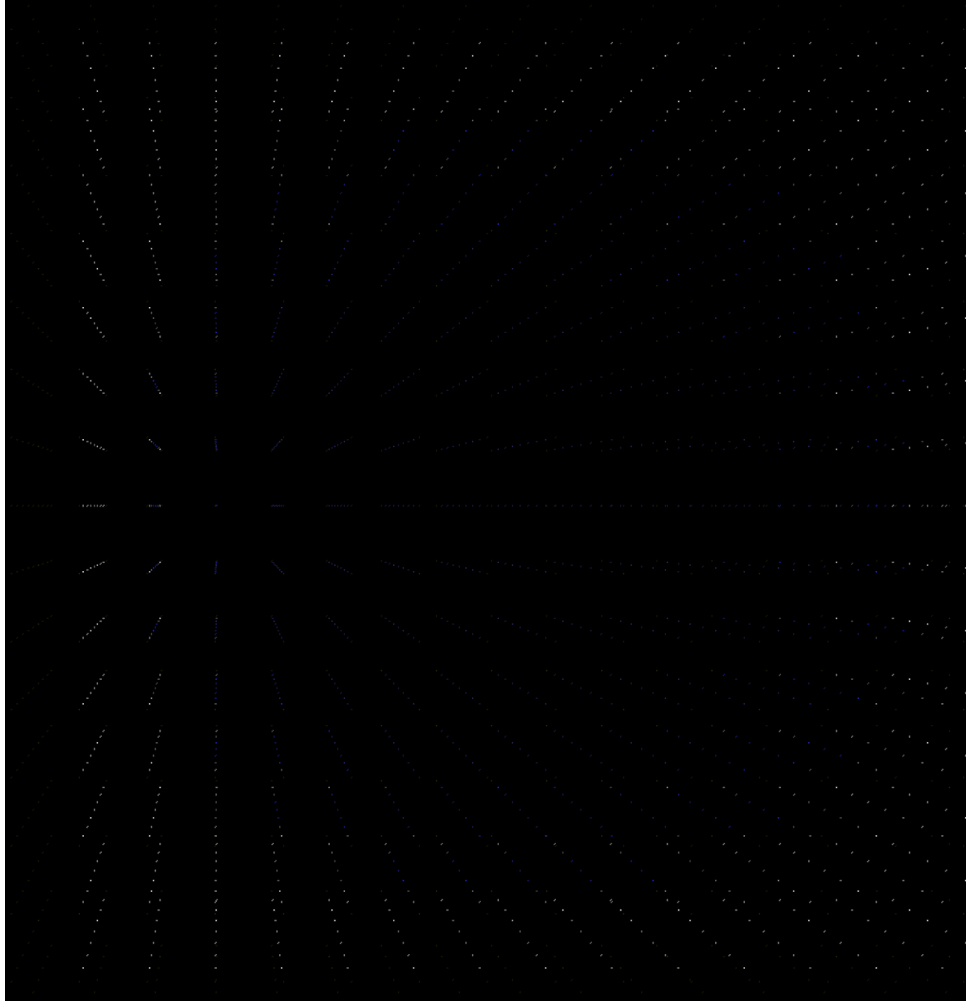
where $\mathbf{r}_j$ is the position of particle j in the coordinate system and $\mathbf{v}_j$ is its velocity, q_j is the 'charge' of particle j , $A_{n,j}$ is the attraction between particle n and particle j and D_j is the damping acting on particle j .

(b) Hodgkin - Huxley (HH) Coupled Neurons

The HH neuron network is based on the voltage potential of each neuron due to chemical reactions taking place at the neuron. Neurons can affect other neurons and neurons can be affected by outside voltage. The HH model is actually a partial differential equation. It is discretized through the neurons to get the following system of differential equations.

$$\begin{aligned}
V'_{i,j,k}(t) &= -\frac{1}{cM_{i,j,k}} (PDEx(V_{i+1,j,k} - 2V_{i,j,k} + V_{i-1,j,k}) + \\
&\quad PDEy(V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k}) + \\
&\quad PDEz(V_{i,j,k+1} - 2V_{i,j,k} + V_{i,j,k-1}) + \\
&\quad \bar{g}_K n_{i,j,k}(t)^4 + \bar{g}_{Na} m_{i,j,k}(t)^3 h_{i,j,k}(t) + \bar{g}_L) V_{i,j,k} + \\
&\quad \frac{1}{cM_{i,j,k}} (\bar{g}_K E_K n_{i,j,k}(t)^4 + \bar{g}_{Na} E_{Na} m_{i,j,k}(t)^3 h_{i,j,k}(t) + \\
&\quad \bar{g}_L E_L) + \frac{I_{M_{i,j,k}}(t)}{cM_{i,j,k}} \\
n'_{i,j,k} &= -(\alpha_n(V_{i,j,k}) + \beta_n(V_{i,j,k})) n_{i,j,k} + \alpha_n(V_{i,j,k}) \\
m'_{i,j,k} &= -(\alpha_m(V_{i,j,k}) + \beta_m(V_{i,j,k})) m_{i,j,k} + \alpha_m(V_{i,j,k}) \\
h'_{i,j,k} &= -(\alpha_h(V_{i,j,k}) + \beta_h(V_{i,j,k})) h_{i,j,k} + \alpha_h(V_{i,j,k})
\end{aligned}$$

where $V_{i,j,k}$ is the voltage potential of the neuron at location (i, j, k) , $n_{i,j,k}$ is the potassium concentration, $m_{i,j,k}$ is the sodium concentration and $h_{i,j,k}$ is other chemical concentrations, $C_{M_{i,j,k}}$ is the neuron's capacitance and $I_{M_{i,j,k}}$ is the applied voltage at the neuron. The other parameters are determined by electro-chemical properties of the neurons involved.



A snapshot of neurons lighting up during a voltage input. The brighter lights have greater voltage potential.

2. PLANETARY MOTION - Newton's N-Body Problem

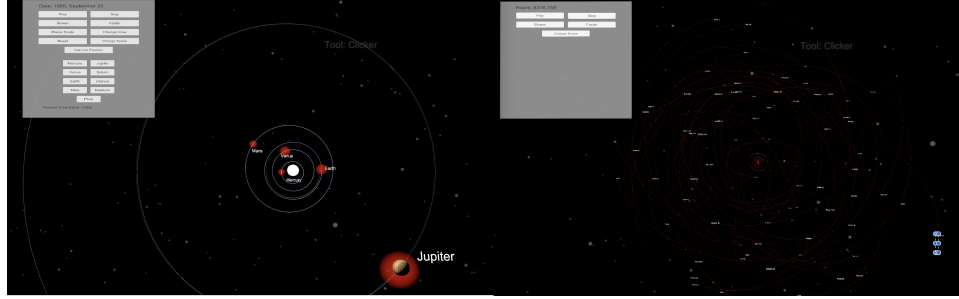
The motion of the planets in a solar system can be modeled by the

differential equations

$$\mathbf{r}'_j(t) = \mathbf{v}_j(t)$$

$$\mathbf{v}'_j(t) = \sum_{n=1}^{j-1} \frac{Gm_n}{\|\mathbf{r}_n - \mathbf{r}_j\|^3} (\mathbf{r}_n - \mathbf{r}_j) + \sum_{n=j+1}^N \frac{Gm_n}{\|\mathbf{r}_n - \mathbf{r}_j\|^3} (\mathbf{r}_n - \mathbf{r}_j)$$

where $\mathbf{r}_j$ is the position of particle j in the coordinate system and $\mathbf{v}_j$ is its velocity, G is the gravitational constant and m_n is the mass of body n . Below are two snapshots of the motion of the planets around the sun and the motion of 72 moons around Jupiter.



3. PRESSURE DISTURBANCES - Euler's Equations

If a disturbance (explosion) is set off in a medium then the pressure exerted at each point in the medium over time can be modeled using Euler's partial differential equations

$$p_t + c^2 \operatorname{div} m = 0$$

$$m_{1t} + p_{x_1} = F_1(x, t)$$

$$m_{2t} + p_{x_2} = F_2(x, t)$$

$$m_{3t} + p_{x_3} = F_3(x, t)$$

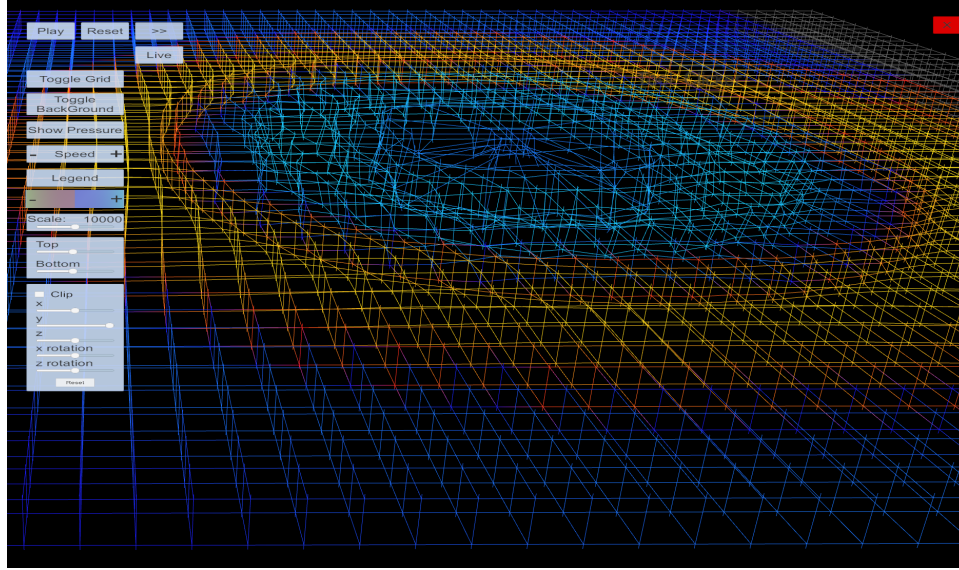
where p is the pressure, m is the momentum of the particle in the medium and F models the explosion. These partial differential equations are discretized in (x_1, x_2, x_3) to generate a system of differential equations on which PSM can be applied.

4. ELASTIC WAVE PROPAGATION - Particle Motion Using Newton's Force Laws

The motion of the particles in an elastic medium can be modeled using the partial differential equations

$$\begin{aligned}\rho u_{tt} &= \frac{\partial}{\partial x} (\lambda(u_x + v_y + w_z) + 2\mu u_x) + \frac{\partial}{\partial y} (\mu(u_y + v_x)) + \frac{\partial}{\partial z} (\mu(w_x + u_z)) \\ \rho v_{tt} &= \frac{\partial}{\partial x} (\mu(u_y + v_x)) + \frac{\partial}{\partial y} (\lambda(u_x + v_y + w_z) + 2\mu v_y) + \frac{\partial}{\partial z} (\mu(w_y + v_z)) \\ \rho w_{tt} &= \frac{\partial}{\partial x} (\mu(w_x + u_z)) + \frac{\partial}{\partial y} (\mu(w_y + v_z)) + \frac{\partial}{\partial z} (\lambda(u_x + v_y + w_z) + 2\mu w_z).\end{aligned}$$

where ρ is the density distribution of the medium, λ and μ are the Lamé' wave propagation parameters, (u, v, w) is the position of the particle that started at (x, y, z) . Again, these partial differential equations are discretized in (x, y, z) to get a system of differential equations for PSM.



A snapshot showing the pressure distribution and the particle displacement in a medium with varying density after an explosion.

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